Catalog

1

Teacher:
Analysis IV - GM

3 hours

SCIPER :

Wait until the start of the exam before turning the first page. This is a two-side printed document, it contains 24 pages, the last ones could be empty. There are 28 questions for a total of 114 points. Do not separate the pages.

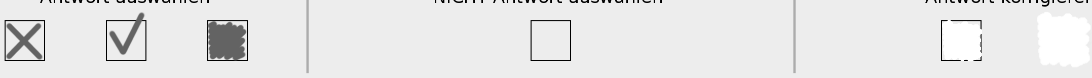
- Put your student card on the table.
- You may use one sheet (A4) with notes on both sides as support.
- The form for Laplace and Fourier transforms is provided.
- Using any **electronic tools** (calculator, telephone, etc.) are prohibited.
- For **single choice questions** you get :
  the indicated number of points if the answer is correct,
  0 points if no answer or more than one answer is given,
  $-\frac{1}{3}$ of the indicated number of points if the answer is wrong.
- For **True-False** questions you get :
  $+1$ point if the answer is correct,
  0 points if there is none answer or more than one answer entered,
  -1 point if the answer is wrong.
- Use a **pen** with **black or dark blue** ink and erase cleanly with **white correction** if necessary.
- If a question turns out to contain an error, the lecturer retains the right to revoke this question.

Respectez les consignes suivantes | Read these guidelines | Beachten Sie bitte die unten stehenden Richtlinien

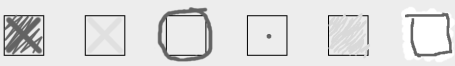
choisir une réponse | select an answer
Antwort auswählen

ne PAS choisir une réponse | NOT select an answer
NICHT Antwort auswählen

Corriger une réponse | Correct an answer
Antwort korrigieren



ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte



Tables

Fourier transform table

For $\omega \in \mathbb{R}$

	$f(y),\ y \in \mathbb{R}$	$\mathcal{F}(f)(\alpha) = \hat{f}(\alpha),\ \alpha \in \mathbb{R}$
1	$f(y) = \begin{cases} 1, & \text{if } \lvert y\rvert < \lvert b\rvert \\ 0, & \text{otherwise} \end{cases}$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\sin(\lvert b\rvert\alpha)}{\alpha}$
2	$f(y) = \begin{cases} 1, & \text{if } b < y < c \\ 0, & \text{otherwise} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-ib\alpha} - e^{-ic\alpha}}{i\alpha}$
3	$f(y) = \begin{cases} e^{-\omega y}, & \text{if } y > 0 \\ 0, & \text{otherwise} \end{cases} \quad (\omega > 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1}{\omega + i\alpha}$
4	$f(y) = \begin{cases} e^{-\omega y}, & \text{if } b < y < c \\ 0, & \text{otherwise} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-(\omega+i\alpha)b} - e^{-(\omega+i\alpha)c}}{\omega + i\alpha}$
5	$f(y) = \begin{cases} e^{-i\omega y}, & \text{if } b < y < c \\ 0, & \text{otherwise} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{i\sqrt{2\pi}} \frac{e^{-i(\omega+\alpha)b} - e^{-i(\omega+\alpha)c}}{\omega + \alpha}$
6	$f(y) = \frac{1}{y^2 + \omega^2} \quad (\omega \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{\pi}{2}} \frac{e^{-\lvert\omega\alpha\rvert}}{\lvert\omega\rvert}$
7	$f(y) = \frac{e^{-\lvert\omega y\rvert}}{\lvert\omega\rvert} \quad (\omega \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2 + \omega^2}$
8	$f(y) = e^{-\omega^2 y^2} \quad (\omega \neq 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2}\lvert\omega\rvert} e^{-\frac{\alpha^2}{4\omega^2}}$
9	$f(y) = ye^{-\omega^2 y^2} \quad (\omega \neq 0)$	$\hat{f}(\alpha) = \frac{-i\alpha}{2\sqrt{2}\lvert\omega\rvert^3} e^{-\frac{\alpha^2}{4\omega^2}}$
10	$f(y) = \frac{4y^2}{(y^2 + \omega^2)^2} \quad (\omega \neq 0)$	$\hat{f}(\alpha) = \sqrt{2\pi} \left(\frac{1}{\lvert\omega\rvert} - \lvert\alpha\rvert\right) e^{-\lvert\omega\alpha\rvert}$

Laplace transform table

For $\alpha > 0$, $\omega \in \mathbb{R}$, $z_0 \in \mathbb{C}$

	$f(t),\ t \geq 0$	$\mathcal{L}(f)(z) = F(z)$	
1	$f_\alpha(t) = \begin{cases} 1/\alpha, & \text{if } t \in [0, \alpha] \\ 0, & \text{otherwise} \end{cases}$	$F_\alpha(z) = \frac{1 - e^{-\alpha z}}{\alpha z} \xrightarrow[\alpha \to 0]{} 1$	$\forall z \in \mathbb{C}$
2	$f(t) = 1$	$F(z) = \frac{1}{z}$	$\mathrm{Re}(z) > 0$
3	$f(t) = e^{-z_0 t}$	$F(z) = \frac{1}{z + z_0}$	$\mathrm{Re}(z + z_0) > 0$
4	$f(t) = \frac{t^n}{n!}$	$F(z) = \frac{1}{z^{n+1}}$	$\mathrm{Re}(z) > 0$
5	$f(t) = te^{-z_0 t}$	$F(z) = \frac{1}{(z + z_0)^2}$	$\mathrm{Re}(z + z_0) > 0$
6	$f(t) = \sin(\omega t)$	$F(z) = \frac{\omega}{z^2 + \omega^2}$	$\mathrm{Re}(z) > 0$
7	$f(t) = \cos(\omega t)$	$F(z) = \frac{z}{z^2 + \omega^2}$	$\mathrm{Re}(z) > 0$
8	$f(t) = e^{z_0 t} \sin(\omega t)$	$F(z) = \frac{\omega}{(z - z_0)^2 + \omega^2}$	$\mathrm{Re}(z - z_0) > 0$
9	$f(t) = e^{z_0 t} \cos(\omega t)$	$F(z) = \frac{z - z_0}{(z - z_0)^2 + \omega^2}$	$\mathrm{Re}(z - z_0) > 0$
10	$f(t) = \sinh(\omega t)$	$F(z) = \frac{\omega}{z^2 - \omega^2}$	$\mathrm{Re}(z) > \lvert\omega\rvert$
11	$f(t) = \cosh(\omega t)$	$F(z) = \frac{z}{z^2 - \omega^2}$	$\mathrm{Re}(z) > \lvert\omega\rvert$
12	$f(t) = e^{z_0 t} \sinh(\omega t)$	$F(z) = \frac{\omega}{(z - z_0)^2 - \omega^2}$	$\mathrm{Re}(z - z_0) > \lvert\omega\rvert$
13	$f(t) = e^{z_0 t} \cosh(\omega t)$	$F(z) = \frac{z - z_0}{(z - z_0)^2 - \omega^2}$	$\mathrm{Re}(z - z_0) > \lvert\omega\rvert$
14	$f(t) = t\cos(\omega t)$	$F(z) = \frac{z^2 - \omega^2}{(z^2 + \omega^2)^2}$	$\mathrm{Re}(z) > 0$

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question [`SCQ-01 - Holomorphe`] **:** *(2 points)* If $f(z) = x + ay + i(bx + cy)$ is holomorphic for $z = x + iy$, then a, b, c satisfy

- ■ $c = 1$ and $a = -b$
- □ $a = 1$ and $c = -b$
- □ $a = -1$ and $c = b$
- □ $a = b = c = 1$

Question [`SCQ-02-Singularités`] **:** *(2 points)* What are the poles of the function

$$f(z) = \frac{z-1}{z^3 + (2i-2)z^2 + (1-4i)z + 2i}\ ?$$

- □ $z = -2i$
- □ $z = -1, 2i$
- ■ $z = 1, -2i$
- □ $f(z)$ has no poles

Question [`SCQ-03-Singularites 2`] **:** *(2 points)* What are the poles of the function

$$f(z) = \frac{z^2-1}{\sin(z+1)}\ ?$$

- □ $z = k\pi, \quad k \in \mathbb{Z}$
- ■ $z = -1 + k\pi, \quad k \in \mathbb{Z} \setminus \{0\}$
- □ $z = -1 + k\pi, \quad k \in \mathbb{Z}$
- □ $f(z)$ is holomorphic

Question [`SCQ-04-Laurent`] **:** *(2 points)* Suppose that

$$Lf(z) = \sum_{-\infty}^{\infty} c_n z^n$$

is the Laurent series expansion of $f(z)$ at $z_0 = 0$. For which of the following functions holds $c_{2n} = 0$ for any integer $n \in \mathbb{Z}$?

- □ $z^2 + 1$
- □ $e^z \cos(z)$
- □ $\tan(z)(z-1)$
- ■ $z^2 \sinh(z)$

Question [`SCQ-05-Laurent 2`] **:** *(4 points)* What is the singular part of the Laurent series expansion of

$$f(z) = \frac{e^{z-i}}{z^2+1}, \qquad \text{at } z_0 = i\ ?$$

- □ $\dfrac{e^i}{z-i}$
- ■ $\dfrac{-i}{2(z-i)}$
- □ $\dfrac{1}{z^2}$
- □ 0

Question [`SCQ-06-Laurent 3`] **:** *(4 points)* What is the singular part of the Laurent series expansion of

$$f(z) = \frac{z+1}{z^3(z^2+1)}, \qquad \text{at } z_0 = 0\ ?$$

- ■ $\frac{1}{z^3} + \frac{1}{z^2} - \frac{1}{z}$
- □ $\frac{-1}{z^2} + \frac{1}{z}$
- □ $\frac{1}{z^3} - \frac{1}{z}$
- □ 0

Question [SCQ-07-Integration 1] **:** *(2 points)* Consider two distinct complex numbers $z_0, z_1 \in \mathbb{C}$. Let Γ_1, Γ_2 be two regular curves parametrised by

$$\gamma_1, \gamma_2 : [0,1] \to \mathbb{C}, \quad \text{avec} \begin{cases} \gamma_1(0) = \gamma_2(0) = z_0, \\ \gamma_1(1) = \gamma_2(1) = z_1. \end{cases}$$

Which of the following statements is true for any holomorphic function $f : \mathbb{C} \to \mathbb{C}$?

- ☐ $\displaystyle\int_{\Gamma_1} f(z)\,dz \;+\; \int_{\Gamma_2} f(z)\,dz \;=\; z_0 + z_1$
- ☐ $\displaystyle\int_{\Gamma_i} f(z)\,dz = (z_1 - z_0) \cdot \mathrm{Long}(\Gamma_i),\ \ i = 1,2$
- ☐ $\displaystyle\int_{\Gamma_1} f(z)\,dz \;+\; \int_{\Gamma_2} f(z)\,dz \;=\; 0$
- ■ $\displaystyle\int_{\Gamma_1} f(z)\,dz \;=\; \int_{\Gamma_2} f(z)\,dz$

Question [SCQ-09-Integration 3] **:** *(2 points)* Let Γ be the unit circle. Which function $f(z)$ satisfies

$$\int_{\Gamma} f(z)\,dz \;=\; 2\pi i \;?$$

- ☐ $f(z) = z^2$
- ☐ $f(z) = \dfrac{1}{z-1}$
- ■ $f(z) = \dfrac{1}{z - 1/2}$
- ☐ $f(z) = \dfrac{1}{(z+\frac{1}{2})^2}$

Question [SCQ-10-Integration 4] **:** *(4 points)* Calculate

$$I = \int_{\gamma} z^2\,dz,$$

where γ is the straight line segment from 1 to i.

- ☐ $I = 0$
- ☐ $I = \dfrac{2i}{3}$
- ■ $I = \dfrac{-1-i}{3}$
- ☐ I is not defined

Question [SCQ-11-Integration 5] **:** *(4 points)* What is the value of the following integral?

$$\int_{|z|=2} \frac{e^{\pi z/2}}{z^2+1}\,dz$$

- ■ $2\pi i$
- ☐ 2π
- ☐ $-\pi$
- ☐ πe

Question [SCQ-12-Residus 1] **:** *(2 points)* What is the residue of the following function $f(z)$ at $z_0 = 1$?

$$f(z) = \frac{z^3}{(z-1)^2(z-2)}.$$

- ☐ -8π
- ☐ $-4i$
- ☐ 0
- ■ -4

Question [`SCQ-13-Laurent 3`] **:** *(2 points)* Which function has the following Laurent series expansion at $z_0 = 0$?

$$f(z) = \sum_{n=-2}^{\infty} (n+1)z^n$$

☐ $\frac{1}{1+z^2}$

☐ $\sin(z)\cos(z)$

■ $\frac{2z-1}{z^4-2z^3+z^2}$

☐ $\frac{z+2}{(z-1)^2}$

Question [`SCQ-14-Residus 3`] **:** *(2 points)* If $f(z)$ has first order poles at $z = 1$ and $z = 2$ with residues 3 and -1, respectively, what is the value of the following integral?

$$\int_{|z-\frac{3}{2}|=\frac{3}{2}} f(z)\,dz$$

☐ $2\pi i$

■ $4\pi i$

☐ 2π

☐ 4π

Question [`SCQ-15-Laplace`] **:** *(2 points)* Which of the following is the Laplace transform of $f(t) = \cos(2t)$?

■ $\frac{z}{z^2+4}$

☐ $\frac{2}{z^2+4}$

☐ $\frac{z}{z^2-4}$

☐ $\frac{2}{z^2-4}$

Question [`SCQ-16-Laplace`] **:** *(4 points)* What is the inverse Laplace transform of

$$F(z) = \frac{1}{z^2+z}, \quad \text{where } \operatorname{Re}(z) > 0\ ?$$

■ $1 - e^{-t}$

☐ $t - e^{-t}$

☐ $e^{-t} - t$

☐ e^{-t}

Question [`SCQ-17-Laplace 2`] **:** *(2 points)* If $F(z)$ is the Laplace transform of $f(t)$, what is the Laplace transform of $t^2 f(t)$?

■ $F''(z)$

☐ $\frac{2}{z}F'(z)$

☐ $z^2 F(z)$

☐ $\int_0^z (z-s)^2 F(s)\,ds$

Question [`SCQ-18-EDP 1`] **:** *(4 points)* Let $\gamma : [0,1] \to \mathbb{C}$ be given by $\gamma(t) = 2e^{it}$. Suppose that $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is holomorphic. If $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is defined by $g(z) = zf'(z)$, what is the residue of f at 0?

■ $\frac{i}{2\pi} \int_\gamma g(z)\,dz$

☐ $\frac{-1}{\pi} \int_\gamma \frac{g(z)}{z}\,dz$

☐ $\frac{1}{4\pi i} \int_\gamma g(z) z\,dz$

☐ $\frac{1}{\pi i} \int_\gamma \frac{g(z)}{z}\,dz$

Question [SCQ-19-PDE 2] : *(6 points)* Consider the following differential equation:

$$\frac{\partial u}{\partial t}(x,t) + (2t+1)\frac{\partial^4 u}{\partial x^4}(x,t) = 0 \text{ for } x \in \mathbb{R},\ t > 0$$

$$u(x,0) = g(x) \text{ for } x \in \mathbb{R},$$

where $g : \mathbb{R} \to \mathbb{R}$. Letting $\hat{g}$ denote the Fourier transform of g, which of the following functions solves the differential equation?

Reminder: the solution to the Cauchy initial-value problem

$$y'(t) + a(t)y(t) = 0, \qquad y(0) = y_0$$

is given by

$$y(t) = y_0 e^{-\int_0^t a(s)\,ds}.$$

- ☐ $u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(\alpha) e^{\alpha^4 t^4} e^{i\alpha x}\, d\alpha$
- ☐ $u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(\alpha) e^{i\alpha^4 t} e^{i\alpha x}\, d\alpha$
- ■ $u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(\alpha) e^{-\alpha^4 (t^2+t)} e^{i\alpha x}\, d\alpha$
- ☐ $u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(\alpha) e^{-\alpha^4 t^2} e^{i\alpha x}\, d\alpha$

Second part: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question [tf-01] : The integral of $f(z) = z^2 + \sin(z^2)$ along every simple regular closed curve $\gamma : [0, 2\pi] \to \mathbb{C}$ is zero.

■ TRUE ☐ FALSE

Question [tf-04] : The function $f(z) = \exp(1/z)$ has a pole of first order at $z_0 = 0$.

☐ TRUE ■ FALSE

Question [tf-06] : The function $f(z) = 1/z$ is holomorphic over $\mathbb{C}$.

☐ TRUE ■ FALSE

Question [tf-08] : The derivative of $f(z) = z\cos(z)$ satisfies the Cauchy-Riemann equations.

■ TRUE ☐ FALSE

Question [tf-09] : The function $f(z) = \mathrm{Re}(z)$, which maps every complex number to its real part, is holomorphic.

☐ TRUE ■ FALSE

Third part, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 24: *8 points*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ■8 *Do not write here.*

Let $f(z) = u(x, y) + iv(x, y)$ be a holomorphic function. Using the Cauchy-Riemann equations, find the derivative $f'(z)$ if

$$u(x, y) = e^{x^2-y^2} \cos(2xy).$$

Question 25: *10 points*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ■ 10 *Do not write here.*

Suppose that $\gamma : [0, 2\pi] \to \mathbb{C}$ is some simple closed regular curve. Given the function

$$f(z) = \frac{\sin(z^2)}{z^3(z - 2i)},$$

what are the possible values of the curve integral $\int_\gamma f$?

Question 26: *12 points*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 10 ☐ 11 ■ 12 *Do not write here.*

Use the residue theorem to compute the following integral:

$$\int_0^{2\pi} (2 + \cos(t))^2 \; dt.$$

Question 27: *11 points*

0 1 2 3 4 5 6 7 8 9 10 11 *Do not write here.*

Consider the ordindary differential equation

$$y''(t) + 2y'(t) + y(t) = t$$

with initial values at $t = 0$:

$$y(0) = 0, \quad y'(0) = 3.$$

(a) Find the Laplace transform $Y(z)$ of the solution y.

(b) Find the solution $y : [0, \infty) \to \mathbb{R}$. You do not need to simplify convolutions.

Recall that the Laplace transformation of a function $f : [0, \infty) \to \mathbb{R}$ is

$$\mathfrak{L}(f)(z) = \int_0^\infty f(t)e^{-zt}\, dt.$$

Question 28: *16 points*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8
☐ 9 ☐ 10 ☐ 11 ☐ 12 ☐ 13 ☐ 14 ☐ 15 ■ 16

Do not write here.

Consider the following partial differential equation over the interval $[0, 1]$:

$$\frac{d^2}{d^2t}u(x,t) - 2\frac{d}{dt}u(x,t) = \frac{d^2}{d^2x}u(x,t), \quad 0 < x < 1, \quad t > 0.$$

Suppose we have Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t > 0,$$

and initial data

$$u(x,0) = 0, \quad 0 < x < 1,$$

$$\frac{d}{dt}u(x,0) = 1, \quad 0 < x < 1.$$

Express the solution $u(x,t)$ in terms of a Fourier sine series

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin(\pi n x).$$

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